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Equal Curvature and Equal Constraint Cantilevers: Extensions of Euler and Clebsch Formulas

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Abstract. The flexure of *cantilevers* is one of the early problems, if not the first, to have been studied by the elasticity theoreticians. One considers axisymmetrical rods and rectangular section beams. This investigation concerns the case where the maximum stress is constant (Galilei–Euler–Clebsch problem) as well as the case where the curvature of the medium fiber is constant. For both cases, it is shown that the equations to solve belong to the same class. The research was into thickness distributions replying to those conditions under various loading cases.

At the free end, the distributions obtained degenerate into a family for which the thickness is null, but contrary to a widely held opinion, they also and naturally give forms showing a *finite thickness* at this end. The proposed distributions have a general form which has not been found in the literature treating elasticity theory or strength of materials [1–9]. They are extensions of Euler and Clebsch formulas.

Sommario. La flessione delle mensole è uno dei primi problemi, se non il primo che sia stato studiato dai teorici di elasticità. Si considerano aste assialsimmetriche e travi a sezione rettangolare. Tale analisi concerne sia il caso in cui la tensione massima è costante (Galilei–Euler–Clebsch problems), sia quello in cui lo è la curvatura della fibra media. Per entrambi i casi si dimostra che le equazioni risolventi appartengono alla medesima classe. Si ricercano le distribuzioni di spessore corrispondenti alle suddette situazioni, in riferimento a varie condizioni di carico.

All'estremo libero le distribuzioni ottenute degenerano in una famiglia per la quale lo spessore è nullo ma, contrariamente ad un'opinione largamente diffusa, esse inoltre forniscono naturalmente delle forme che presentano uno *spessore finito* a tale estremo. Le distribuzioni proposte hanno una forma generale che non è presente nella letteratura tecnica concernente la teoria dell'elasticità o la resistenza dei materiali [1–9]. Esse costituiscono alcune estensioni delle formule di Eulero e Clebsh.

Key words: Elasticity, Equal curvature, Equal constraint, Structural mechanics.

1. Introduction

The flexure of cantilevers is one of the early problems that was investigated by the elasticity theoreticians. We owe the determination of some thickness distributions of *equal strength* beams to Euler and to Clebsch, about which Galilei had foreseen a solution [1–9]. The thickness distributions $h(z)$ of bars having a *constant curvature* are researched in order to compare them with those providing an equal strength.

Let us consider bars of length l, having a built-in end at the origin $z = 0$. The axis of symmetry of these bars is Oz, the other horizontal axis being Ox. The flexure $w(z)$ is in the vertical direction, i.e. Oy axis. Denoting E as the Young modulus, the local curvature $1/r$ at a point of z abscissa is linked to the maximal stress σ located at the bar surface, i.e. at a distance $h/2$ from the neutal fiber, by the relationship

$$\frac{1}{r(z)} = \frac{2}{E}\frac{\sigma(z)}{h(z)}, \tag{1}$$

I_x being the inertia moment with respect to Ox. The flexural moment is

$$M = EI_x \frac{d^2 w}{dz^2} = EI_x \frac{1}{r} = 2I_x \frac{\sigma}{h}. \tag{2}$$

Denoting P the weight of a rod, the shearing force $T(z)$ is of opposite direction to the sum of external forces applied onto the $[z, l\,]$ segment

$$T(z) = -P + \frac{\pi}{4}\mu g \int_0^z h^2\, dz = -\frac{\pi}{4}\mu g \int_z^l h^2\, dz. \tag{3}$$

Similarly, for a beam of constant width a in the direction Ox and of thickness $h(z)$ in the vertical plane, the shearing forces due to own weight, to a force F vertically applied onto the free end or to a uniform load f by length unit applied onto all the beam, are respectively

$$T(z) = -\mu g a \int_z^l h\, dz; \qquad -F; \qquad -(l-z)f. \tag{4}$$

The equilibrium relationship for these bars is given by

$$\frac{dM}{dz} - T(z) = 0. \tag{5}$$

With these notations, the inertia moments of a rod of h diameter and of a beam of a width are respectively

$$I_x = \pi h^4/64, \qquad I_x = ah^3/12. \tag{6}$$

For bars which maximal stress σ is a constant all along its surface, one has to solve equation (5), i.e.

$$2\sigma \frac{d}{dz}\left(\frac{I_x}{h}\right) - T(z) = 0. \tag{7a}$$

For bars which the deformation curve shows a constant curvature $1/r$ (i.e. a *parabola*), one has to solve

$$\frac{E}{r}\frac{d}{dz} I_x - T(z) = 0. \tag{7b}$$

Let us consider hereafter various cases where one of the ends is built-in. The geometrical representation of the thickness profile is denoted $h(0) = h_0$ at the built-in end and $h(l) = h_l$ at the free end.

2. Bars Bent by a Force F at the Free End

All these bars belong to a thickness class defined by the solutions of a differential equation, issued from equation (7a) or (7b), whose type is

$$h^p \frac{dh}{dz} + \alpha = 0, \tag{8}$$

where p is an integer. These solutions are represented by the general form

$$h(z) = h_0 \left\{ 1 - \left[1 - \left(\frac{h_l}{h_0} \right)^{p+1} \right] \frac{z}{l} \right\}^{1/(p+1)} . \tag{9a}$$

The thickness h_l is defined by the relationship

$$1 - \left(\frac{h_l}{h_0} \right)^{p+1} = (p+1) \frac{\alpha l}{h_0^{p+1}} . \tag{10}$$

One remarks that h_l remains finite. For $h_l = 0$, the distribution given by equation (9a) degenerates and simply becomes

$$h(z) = h_0 \left\{ 1 - \frac{z}{l} \right\}^{1/(p+1)} . \tag{9b}$$

As an example, let us consider the cases of rods of circular sections and beams of rectangular sections having a constant width, bent according to a constant stress σ or a constant curvature $1/r$. This leads to the study of four cases.

2.1. Rod Flexed by a Force F with Constant Constraint

Let us consider constant constraint rods deformed by a vertical force applied onto the end $x = l$. In this case, the shearing force is a constant $T = -F$. The differential equation of the deformation is

$$h^2 \frac{\mathrm{d}h}{\mathrm{d}z} + \frac{32}{3\pi} \frac{F}{\sigma} = 0, \tag{11}$$

i.e. $p = 2$. The solution

$$h = h_0 \left\{ 1 - [1 - (h_l/h_0)^3] \frac{z}{l} \right\}^{1/3}, \quad \text{with } 1 - \left(\frac{h_l}{h_0} \right)^3 = \frac{32}{\pi} \frac{Fl}{\sigma h_0^3}, \tag{12}$$

is a *section of cubic parabola* having its vertex at $z_s = l/(1 - h_l^3/h_0^3)$ and slope at the origin

$$\left[\frac{\mathrm{d}h}{\mathrm{d}z} \right]_{z=0} = -\frac{h_0}{3l} \left[1 - \left(\frac{h_l}{h_0} \right)^3 \right] .$$

The flexure from equation (2) is

$$w = \frac{9\sigma l^2 h_0^5}{5E(h_0^3 - h_l^3)^2} \left\{ \left[1 - (1 - h_l^3/h_0^3) \frac{z}{l} \right]^{5/3} + \frac{5}{3} (1 - h_l^3/h_0^3) \frac{z}{l} - 1 \right\} . \tag{13}$$

2.2. Beam Flexed by a Force F with Constant Constraint

Let us consider a beam of vertical thickness $h(r)$ and of width a in the direction of the Ox axis, deformed by a vertical force F applied at its end $x = l$. The shearing force $T = -F$ is a constant. From equations (2) and (6), $M = a\sigma h^2/6$. The differential equation of the deformation is

$$h \frac{\mathrm{d}h}{\mathrm{d}z} + \frac{3F}{a\sigma} = 0, \tag{14}$$

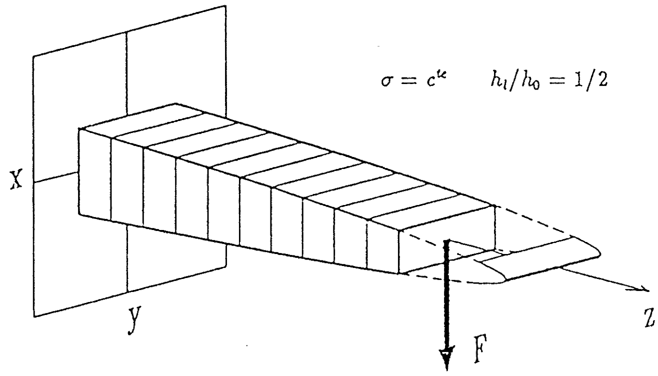

Figure 1. The thickness distribution of a beam flexing under a constant constraint is a *truncated parabola*.

i.e. $p = 1$, and the solution

$$h = h_0 \left\{ 1 - [1 - (h_l/h_0)^2]\frac{z}{l} \right\}^{1/2}, \quad \text{with } 1 - \left(\frac{h_l}{h_0}\right)^2 = \frac{6Fl}{\sigma a h_0^2}, \tag{15}$$

is a *parabola section* (cf. Figure 1). Equation (2) gives the deformation by taking into account $w(0) = 0$ and $[\mathrm{d}w/\mathrm{d}z]_{z=0} = 0$, thus

$$w = \frac{8\sigma l^2 h_0^3}{3E(h_0^2 - h_l^2)^2} \left\{ \left[1 - (1 - h_l^2/h_0^2)\frac{z}{l} \right]^{3/2} + \frac{3}{2}(1 - h_l^2/h_0^2)\frac{z}{l} - 1 \right\}. \tag{16}$$

2.3. Rod Flexed by a Force F with Constant Curvature

The vertical force F applied at the end $z = l$ generates a shearing force $T = -F$. The differential equation of the thickness is

$$h^3 \frac{\mathrm{d}h}{\mathrm{d}z} + \frac{16}{\pi}\frac{Fr}{E} = 0, \tag{17}$$

i.e. $p = 3$, and the solution

$$h = h_0 \left\{ 1 - \left[1 - \left(\frac{h_l}{h_0}\right)^4 \right] \frac{z}{l} \right\}^{1/4}, \quad \text{with } 1 - \left(\frac{h_l}{h_0}\right)^4 = \frac{64Frl}{\pi E h_0^4}, \tag{18}$$

is the *section of a biquadratic* having a null axial curvature.

2.4. Beam Flexed by a Force F with Constant Curvature

The differential equation of the thickness is

$$h^2 \frac{\mathrm{d}h}{\mathrm{d}z} + 4\frac{Fr}{Ea} = 0, \tag{19}$$

i.e. $p = 2$, and the solution

$$h = h_0 \left\{ 1 - \left[1 - \left(\frac{h_l}{h_0}\right)^3 \right] \frac{z}{l} \right\}^{1/3}, \quad \text{with } 1 - \left(\frac{h_l}{h_o}\right)^3 = \frac{12Frl}{Eah_0^3}, \tag{20}$$

is a *section of cubic parabola*. This form is identical to that obtained in Section 2.1.

For these constant curvature bars, the equation of deformation is obviously denoted $w = (1/2r)z^2$ with $\sigma = (E/2r)h(z)$.

3. Bars Deformed by a Uniform Line Load f

Let us consider bars bent by a uniform line load f vertically applied all along the length. For the previously studied cases, relationships (7a) and (7b) lead to

- rods with $\sigma =$ constant $\quad h^2 \frac{dh}{dz} + \frac{32f}{3\pi\sigma}(l - z) = 0,$
- beams with $\sigma =$ constant $\quad h \frac{dh}{dz} + \frac{2f}{a\sigma}(l - z) = 0,$
- rods with $r =$ constant $\quad h^3 \frac{dh}{dz} + \frac{16rf}{\pi E}(l - z) = 0,$
- beams with $r =$ constant $\quad h^2 \frac{dh}{dz} + \frac{4rf}{aE}(l - z) = 0.$

These bars belong to a thickness class solution of the equation

$$h^p \frac{dh}{dz} + \beta(l - z) = 0, \tag{21}$$

coming from

$$\frac{d}{dz}\left(h^p \frac{dh}{dz}\right) - \beta = 0, \tag{22}$$

where p is an integer and β a coefficient having the dimension of $p - 1$. The separable variables provide the solution of equation (21)

$$\frac{1}{p+1} h^{p+1} = \frac{1}{2}\beta(l - z)^2 + C_1.$$

The constant is determined by $h = h_l$ for l=0 thus $C_1 = h_l^{p+1}/(p + 1)$, which allows to define h_l from β and h_0

$$1 - \left(\frac{h_l}{h_0}\right)^{p+1} = \frac{p+1}{2} \frac{\beta l^2}{h_0^{p+1}}. \tag{23}$$

By substitution, one obtains

$$h^{p+1} - h_l^{p+1} = (h_0^{p+1} - h_l^{p+1})\left(1 - \frac{z}{l}\right)^2, \tag{24}$$

and finally the studied distributions to be compared with equation (9a)

$$h(z) = h_0 \left\{1 - \left[1 - \left(\frac{h_l}{h_0}\right)^{p+1}\right]\left(2 - \frac{z}{l}\right)\frac{z}{l}\right\}^{1/(p+1)}. \tag{25a}$$

For h_l=0, they degenerate and become more simply

$$h(z) = h_0 \left\{1 - \frac{z}{l}\right\}^{2/(p+1)}, \tag{25b}$$

to be compared to (9b). They give a *triangular* distribution with *cutting edge* for the beams at constant constraint ($p = 1$) and also free ends with vertical tangents, i.e. a *rounded edge* for the three other cases ($p = 2$ and $p = 3$).

4. Bars Deformed Under Their Own Weight

One considers bars flexing under their own weight. With the treated examples, the starting equations, issues from (7a) and (7b), are as follows:

- rods with $\sigma =$ constant $\quad \frac{\mathrm{d}}{\mathrm{d}z}\left(h^2 \frac{\mathrm{d}h}{\mathrm{d}z}\right) - \frac{8\mu g}{3\sigma} h^2 = 0,$
- beams with $\sigma =$ constant $\quad \frac{\mathrm{d}}{\mathrm{d}z}\left(h \frac{\mathrm{d}h}{\mathrm{d}z}\right) - \frac{3\mu g}{\sigma} h = 0,$
- rods with $r =$ constant $\quad \frac{\mathrm{d}}{\mathrm{d}z}\left(h^3 \frac{\mathrm{d}h}{\mathrm{d}z}\right) - \frac{4\mu g r}{E} h^2 = 0,$
- beams with $r =$ constant $\quad \frac{\mathrm{d}}{\mathrm{d}z}\left(h^2 \frac{\mathrm{d}h}{\mathrm{d}z}\right) - \frac{4\mu g r}{E} h = 0.$

These bars belong to a thickness class solution of the second-order equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(h^p \frac{\mathrm{d}h}{\mathrm{d}z}\right) - \beta h^q = 0, \tag{26}$$

where p and q are integers and β a coefficient having the dimension $p - q - 1$. This class contains that of the previous section for $q = 0$. Denoting H such as [10]

$$H = h^{p+1}/(p+1), \tag{27}$$

one obtains

$$\frac{\mathrm{d}^2 H}{\mathrm{d}z^2} - \beta(p+1)^{q/(p+1)} H^{q/(p+1)} = 0,$$

thus, multiplying by $2\mathrm{d}H/\mathrm{d}z$,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}H}{\mathrm{d}z}\right)^2 - 2\beta(p+1)^{q/(p+1)} H^{q/(p+1)} \frac{\mathrm{d}H}{\mathrm{d}z} = 0.$$

The integration leads to

$$\left(\frac{\mathrm{d}H}{\mathrm{d}z}\right)^2 = \frac{2\beta}{p+q+1}(p+1)^{(p+q+1)/(p+1)} H^{(p+q+1)/(p+1)} + C_1,$$

and coming to $h(z)$

$$\left(\frac{\mathrm{d}h}{\mathrm{d}z}\right)^2 = \frac{2\beta}{p+q+1} h^{q-p+1} + C_1 h^{-2p},$$

so that to integrate

$$z = \int \frac{h^p \mathrm{d}h}{\left[\frac{2\beta}{p+q+1} h^{p+q+1} + C_1\right]^{1/2}}. \tag{28}$$

If one denotes $C_1 = -2\beta h_l^{p+1}/(p+1)$ for $q = 0$, one recovers the relations (23) and (24) of the previous section. By analogy, one can define h_l such as:

$$C_1 = -2\beta h_l^{p+q+1}/(p+q+1). \tag{29}$$

The general solution of equation (26) must satisfy

$$z = \left(\frac{p+q+1}{2\beta}\right)^{1/2} \int \frac{h^p \mathrm{d}h}{[h^{p+q+1} - h_l^{p+q+1}]^{1/2}}. \tag{30}$$

In the practical examples previously considered, the value of integers p and q are:

- rod σ = constant, $p = 2$ and $q = 2$, $p + q + 1 = 5$
- beam σ = constant, $p = 1$ and $q = 1$, $p + q + 1 = 3$
- rod r = constant, $p = 3$ and $q = 2$, $p + q + 1 = 6$
- beam r = constant, $p = 2$ and $q = 1$, $p + q + 1 = 4$

and the integration is also not possible for these integers since $q \neq p + 1$.

In counterpart, it is possible to obtain the distributions of bars having a null thickness at their free edge, i.e. $C_1 = 0$:

$$\frac{z}{l} + C_2 = \frac{2}{p-q+1}\left(\frac{p+q+1}{2\beta l^2}\right)^{1/2} h^{(p-q+1)/2}.$$

Since $h = 0$ for $z = l$, one deduces that $C_2 = -1$ and

$$h^{p-q+1} = \frac{2\beta l^2}{p+q+1}\left(\frac{p-q+1}{2}\right)^2 \left\{1 - \frac{z}{l}\right\}^2.$$

This relationship allows defining the thickness at the origin h_0 by

$$h_0^{p-q+1} = \frac{2\beta l^2}{p+q+1}\left(\frac{p-q+1}{2}\right)^2. \tag{31}$$

The distributions studied in the particular case where the thickness is null at the free end are finally

$$h(z) = h_0 \left\{1 - \frac{z}{l}\right\}^{2/(p-q+1)}. \tag{32}$$

This relationship is more general than equation (25b).

Returning to the general case, taking into account equation (29), one has

$$\left(\frac{\mathrm{d}h}{\mathrm{d}z}\right)^2 = \frac{2\beta}{p+q+1}(h^{p+q+1} - h_l^{p+q+1})h^{-2p}.$$

Let us express this equation under adimensional form. From equation (31), $\beta \propto h_0^{p-q+1} l^{-2}$ of dimension $-(q - p + 1)$. This allows definining h_0 in the general case such as

$$h_0^{p-q+1} = \frac{1}{G^2}\frac{2\beta l^2}{p+q+1}\left(\frac{p-q+1}{2}\right)^2, \tag{33}$$

where G is an adimensional coefficient which takes a unity value for the case of bars having a null thickness at the free end. Denoting $h_{0,0}$ the thickness of these later at the built-in zone, as given by equation (31), one can define G by the relationship

$$G^2 = (h_{0,0}/h_0)^{p-q+1}. \tag{34}$$

So that the differential equation takes the adimensional form

$$\left(\frac{l}{h_0}\frac{\mathrm{d}h}{\mathrm{d}z}\right)^2 = \left(\frac{2}{p-q+1}\right)^2 \left(\frac{h_{0,0}}{h_0}\right)^{p-q+1} \left[\left(\frac{h}{h_0}\right)^{p+q+1} - \left(\frac{h_l}{h_0}\right)^{p+q+1}\right] \left(\frac{h}{h_0}\right)^{-2p}. \tag{35}$$

Finally, by using reduced variables h/h_0 and z/l and by choosing the negative sign in order to have $\mathrm{d}(h/h_0)$ negative so that this is convenient for the particular case $h_l = 0$ previously treated

$$\mathrm{d}\left(\frac{h}{h_0}\right) = -\frac{2}{p-q+1} \left(\frac{h_{0,0}}{h_0}\right)^{\frac{p-q+1}{2}} \left[\left(\frac{h}{h_0}\right)^{p+q+1} - \left(\frac{h_l}{h_0}\right)^{p+q+1}\right]^{\frac{1}{2}} \left(\frac{h}{h_0}\right)^{-p} \mathrm{d}\left(\frac{z}{l}\right). \tag{36}$$

A numerical integration allows to determine the $h(z)/h_0$ distributions as function of the two constants. For this, one has used the inverse of the overthickness ratio $G^{2/(p-q+1)}$ to be determined as a function of the thickness ratio h_l/h_0, this later being given at the free end.

This resolution has been carried out starting from h_l/h_0 ratios contained between 0 and 1, and by finding $h_{0,0}/h_0$ ratios so that $h(z)/h_0$ becomes equal to h_l/h_0 for $z = l$ [cf. Table 1 and Figure 2].

In the four cases considered, the slope of the distributions obtained are somewhat similar to those given by the relationship (32) with h_l=0. With the case of constant curvature rods, where the solution for h_l=0 is a *cone*, one can notice in Figure 3 that the distribution is a *quasi-truncated cone*.

5. Conclusions

The solutions described include the case of bars for which the free end is at null thickness and vertical tangent such as found in some examples in the literature [2–9]. They also include bars for which the free end is a *cutting edge* or a *pointed edge*. In the general case, the distributions that have been found with equations (9a) and (25a) generate a *thickness remaining finite* at the free end.

The expeditious formulation existing in most of the works dealing with these questions [cf. Clebsch–*Annotée*, Timoshenko] is *incorrect* since it consists in stating that, to avoid a null thickness at the free end, one has to make a local overthickness here in such manner that the shearing force will not break the bar. This study demonstrates that, in all cases, the uniconstant theory provides by itself a satisfying answer.

With the case of bars deformed by a force F at the free end, one finds *identical* distributions for constant constraint rods and constant curvature beams. This result applies also to the case of a line force f. By symmetrizing those cantilevers with respect to their built-in plane, thus doubling their length, the present formulas apply. For sufficiently wide plates, i.e. x direction, these results could be applied to obtain cylindrical mirrors deformed from a plane. This would allow the compensation of astigmatism aberration due to tilted collimator or camera mirrors as often used in astronomical spectrographs.

It can be of interest to go from the one-dimensional problem of bars deformed by such external forces to the two-dimensional problem. The cases of a circular plate bent by a central

Table 1. $h_{0,0}/h_0$ ratios as function of h_l/h_0

h_l/h_0	$p = 2, q = 2$	$p = 1, q = 1$	$p = 3, q = 2$	$p = 2, q = 1$
0.000	1.0000	1.0000	1.0000	1.0000
0.025	0.7475	0.7770	0.9810	0.9847
0.050	0.6539	0.6932	0.9623	0.9697
0.100	0.5323	0.5827	0.9249	0.9397
0.150	0.4475	0.5041	0.8876	0.9096
0.200	0.3816	0.4420	0.8502	0.8795
0.250	0.3280	0.3902	0.8129	0.8492
0.300	0.2829	0.3456	0.7755	0.8195
0.400	0.2109	0.2713	0.7005	0.7557
0.500	0.1554	0.2103	0.6247	0.6895
0.600	0.1115	0.1583	0.5468	0.6176
0.700	0.0758	0.1127	0.4647	0.5368
0.750	0.0604	0.0917	0.4206	0.4912
0.800	0.0463	0.0717	0.3733	0.4405
0.850	0.0334	0.0527	0.3211	0.3827
0.900	0.0214	0.0344	0.2606	0.3136
0.950	0.0103	0.0169	0.1833	0.2225
0.975	0.0050	0.0084	0.1293	0.1577
1.000	0.0000	0.0000	0.0000	0.0000

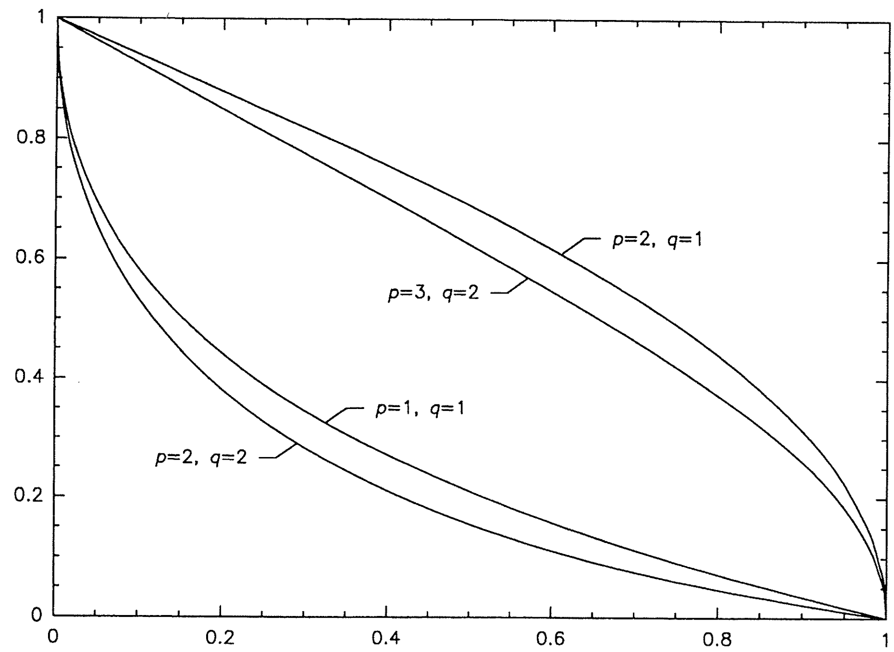

Figure 2. Representation of $h_{0,0}/h_0$ as function of h_l/h_0. For intermediate values of h_l/h_0, the distributions giving constant curvatures (top) vary less than those giving constant constraints (down).

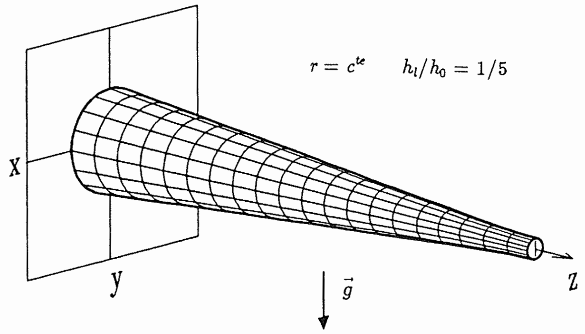

Figure 3. Thickness distribution of a rod flexing under its own weight according to a constant curvature (case $p = 3$ and $q = 2$) and for $h_l/h_0 = 1/5$. The form differs slightly from that of a truncated cone. One finds $G = h_{0,0}/h_0 = 0.8129$.

force, by a uniform load or by both has been investigated by the author who has found variable curvature mirrors and aberration corrected active mirrors [11–13]. These mirrors are useful to astronomy [14–15].

With the case of bars deformed by their own weight, the formula (32) provides all the possible distributions for a null thickness end. When this thickness end remains finite, the resulting integration is displayed by Table 1. This shows that, for h_l/h_0 intermediate values to the [0,1] domain, the *constant curvature* distributions vary much less than the *constant constraint* ones such as displayed by Figure 2.

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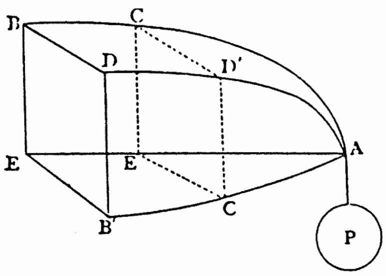
Appendix. Notes to the References

Reference [1]: Galilei was the first to be interested in the flexure of cantilevers and discovered the notion of equal strength solids.



Scheme of an equal strength cantilever by Galilei from *Discorsi*.

Reference [2]: Euler studied the flexure of equal resistance cantilevers having a *flat form*, i.e. a constant vertical thickness h and a variable width $a(z)$ deformed by a vertical force F applied onto the free end. Thus, Euler discovered that $a(z)$ linearly decreases to provide a vertical *cutting edge* at the end $z{=}l$. Starting from equation (7), one can check that this *triangle* distribution is also the correct solution of the equal curvature problem. When the horizontal thickness remains finite at $z{=}l$, I have found that this distribution becomes a *trapezoid* expressed as $a(z) = a_0[1-(1-a_l/a_0)z/l]$. The width difference of the ends is $a_0 - a_l = 6Fl/h^2\sigma = 12Fl\,r/Eh^3$. Let z_s be the abscissa of the intersection point of the lateral sides onto Oz. It is then shown that $z_s/a_0 = h^2\sigma/6F = Eh^3/12F\,r = constant$ i.e. *the width variation of the trapezoid remains a constant which is not dependent on the width a_l at the free end.*

Reference [3]: Clebsch gave the distributions of constant constraint rods having edge thicknesses $h_l = 0$ for two deformation cases: (i) flexed under their own weight $p = q = 2$ (cf. Section 4) where he found a *parabola* and (ii) flexed by a vertical force $p = 2$ where he found a *cubic parabola* (cf. Section 2) (cf. [5; pp. 857–858] and [6; pp. 163–164]).